Ilias S. Kotsireas
Eugene V. Zima
Editors

$$\mathrm{per}\ (A) \geq \frac{n!}{n^n}$$

Advances in Combinatorial Mathematics

Springer

Advances in Combinatorial Mathematics

Ilias S. Kotsireas • Eugene V. Zima
Editors

Advances in Combinatorial Mathematics

Proceedings of the Waterloo Workshop in Computer Algebra 2008

Springer

Editors
Ilias S. Kotsireas
Eugene V. Zima
Wilfrid Laurier University
Department of Physics and Computer Science
75 University Avenue West
Waterloo, Ontario N2L 3C5
Canada
ikotsire@wlu.ca
ezima@wlu.ca

ISBN 978-3-642-03561-6 e-ISBN 978-3-642-03562-3
DOI 10.1007/978-3-642-03562-3
Springer Heidelberg Dordrecht London New York

Library of Congress Control Number: 2009938633

Mathematics Subject Classification (2000): 05-xx, 15-xx

Cover design: WMXDesign GmbH, Heidelberg

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

This book is dedicated to the 70th birthday of Georgy Egorychev, the author of the influential, milestone book "Integral Representation and the Computation of Combinatorial Sums", and a recipient of the Fulkerson Prize for solving the van der Waerden conjecture on the determination of the minimum of the permanent of a doubly stochastic matrix.

Foreword

It is a pleasure for me to have the opportunity to write the foreword to this volume, which is dedicated to Professor Georgy Egorychev on the occasion of his seventieth birthday. I have learned a great deal from his creative and important work, as has the whole world of mathematics. From his life's work (so far) in having made distinguished contributions to fields as diverse as the theory of permanents, Lie groups, combinatorial identities, the Jacobian conjecture, etc., let me comment on just two of the most important of his research areas.

The *permanent* of an $n \times n$ matrix A is

$$\mathrm{Per}(A) = \sum a_{1,i_1} a_{2,i_2} \ldots a_{n,i_n}, \tag{1}$$

extended over the $n!$ permutations $\{i_1, \ldots, i_n\}$ of $\{1, 2, \ldots, n\}$. Thus, the permanent is "like the determinant except for dropping the sign factors from the terms." However by dropping those signs, one loses almost all of the friendly characteristics of determinants, such as the fact that $\det(AB) = \det(A)\det(B)$, the invariance under elementary row and column operations, and so forth. The permanent is a creature of multilinear algebra, rather than of linear algebra, and is much crankier to deal with in virtually all of its aspects, both theoretical and algorithmic.

Nonetheless the permanent is quite an important concept, for example in combinatorial mathematics. The permanent of a matrix whose entries are all either 0's or 1's is (see (1) above) the number of permutations of n letters for which all n of the entries $\{a_{\nu,i_\nu}\}_{\nu=1}^{n}$ are 1's, and this is a valuable tool for counting permutations with restricted positions, for counting Latin rectangles and squares, and so forth.

In 1926, B. L. van der Waerden conjectured that among all $n \times n$ matrices whose entries are nonnegative real numbers and whose row and column sums are all equal to 1, the matrix whose permanent is as small as possible is uniquely the one whose entries are all equal to $1/n$. In view of the numerous applications of permanents, the truth of this conjecture would have valuable consequences. Fifty-six years later the conjecture was proved [1] by Egorychev. (Another proof, found almost simultaneously, is due to Falikman [4].)

That achievement alone would have been enough to assure Professor Egorychev's place on the honor roll of great mathematicians, but we must mention another aspect of his research that reinforces this evaluation. I am referring to his work on combinatorial identities, as described in his book ([2], [3]). There he has shown how a wide class of combinatorial identities can be proved and/or discovered by the methods of complex analysis, thereby making an important contribution to the unity of a subject which has in the past been highly fragmented, but which now, thanks to his and other remarkable advances, is starting to show signs of maturity. Professor Egorychev discusses some recent developments of this line of thought in Chapter 1 below.

As you read the contributions by his friends and colleagues in this volume, take note of the variety and the beauty of the fields of mathematics that they encompass, and reflect on the varied and extensive advances in mathematics that we owe to Professor Georgy Egorychev.

References

[1] Егорычев, Г. П. [Egorychev, G. P.] Решение проблемы ван-дер-Вардена для перманентов. (Russian) [Solution of the van der Waerden problem for permanents] Preprint IFSO-13M. *Akad. Nauk SSSR Sibirsk. Otdel., Inst. Fiz.*, Krasnoyarsk, 1980. 12 pp.

[2] Егорычев, Г. П. [Egorychev, G. P.], Интегральное представление и вычисление комбинаторных сумм. (Russian) [Integral representation and computation of combinatorial sums] Izdat. "Nauka" Sibirsk. Otdel., Novosibirsk, 1977. 283 pp.

[3] Egorychev, G. P. Integral representation and the computation of combinatorial sums. Translated from the Russian by H. H. McFadden. Translation edited by Lev J. Leifman. Translations of Mathematical Monographs, **59**. American Mathematical Society, Providence, RI, 1984. x+286 pp.

[4] Фаликман Д. И. [Falikman, D. I.], Доказательство гипотезы ван дер Вардена о перманенте дважды стохастическои матрицы [Proof of the van der Waerden conjecture on the permanent of a doubly stochastic matrix.] (Russian) *Mat. Zametki* 29 (1981), no. 6, 931–938, 957.

Philadelphia, PA, USA April 23, 2009 *Herbert S. Wilf*

Preface

The Second Waterloo Workshop on Computer Algebra (WWCA 2008) was held May 5-7, 2008 at Wilfrid Laurier University, Waterloo, Canada. This conference was dedicated to the 70$^{\text{th}}$ birthday of Georgy Egorychev (Krasnoyarsk, Russia), who is well known and highly regarded as the author of the influential, milestone book "Integral Representation and the Computation of Combinatorial Sums," which described a regular approach to combinatorial summation, today also known as the method of coefficients. Another great success of this Russian mathematician came in 1980, when he solved the van der Waerden conjecture on the determination of the minimum of the permanent of a doubly stochastic matrix and was awarded the D.R. Fulkerson Prize.

Topics discussed at the workshop[1] were devoted to these two themes (combinatorial and algorithmic summation and special polynomials) and related problems in enumerative combinatorics. The workshop's format included invited lectures and presentations, and it attracted international participants from the USA, Europe, Taiwan, as well as several Canadian universities. Different aspects of the method of coefficients and its relation to algorithmic summation methods and methods of proving combinatorial identities were thoroughly discussed by George E. Andrews (Pennsylvania State University, USA), Georgy Egorychev (Siberian Federal University, Russia), Ira Gessel (Brandeis University, USA), I-Chiau Huang (Institute of Mathematics, Taiwan), Peter Paule (RISC-Linz, Austria), Marko Petkovsek (University of Ljubljana, Slovenia), and Doron Zeilberger (Rutgers University, USA). The theory and applications of the permanent and other special polynomials were presented by Leonid Gurvits (Los Alamos National Laboratory, USA) and Herbert Wilf (University of Pennsylvania, USA). Michiel Hazewinkel (CWI, the Netherlands) discussed the "niceness" of mathematical objects and theorems.

The workshop was financially supported by the Fields Institute of the University of Toronto and various offices of Wilfrid Laurier University.

[1] http://www.cargo.wlu.ca/wwca2008/

This book presents a collection of selected formally refereed papers submitted after the workshop. The topics discussed in this book are closely related to Georgy Egorychev's influential works.

This book would not have been possible without the dedication and hard work of the anonymous referees, who supplied detailed referee reports and helped authors to improve their papers significantly. Finally, we wish to thank the people at Springer-Verlag, in particular Ruth Allewelt and Martin Peters, for working closely with us and for their unequivocal support throughout the entire publication process.

Waterloo,
May 2009

Ilias S. Kotsireas
Eugene V. Zima

Contents

Chapter 1
Method of Coefficients: an algebraic characterization and recent applications

Georgy P. Egorychev

Abstract The article is devoted to the algebraic-logical foundations of the analytic approach to summation problems in various fields of mathematics and its applications. Here we present the foundations of the method of coefficients developed by the author in late 1970's and its recent applications to several well-known problems.

1.1 Introduction

The article is devoted to the algebraic-logical foundations of the analytical approach to summation problems in various fields of mathematics and its applications. Here we present the foundations of the method of integral representations and computation of combinatorial sums (the method of coefficients) developed by the author in the end of 1970's [25] and its recent applications to several well-known problems (see reviews in [26, 31]). The article contains several new results, including the method of coefficients (the set of inference rules and the Completeness Lemma) with operations in the ring of formal Dirichlet series of usual type, as well as several new properties of the characteristic function of the stopping height for the Collatz problem [27, 28], and the solutions of two interesting problems of summation in the theory of holomorphic functions in $\mathbb{C}^n$. Finally we shall give a new algebraic characterization of the method of coefficients, which is based on the φ-*operation of isomorphism* [9, 20, 59], generated by the classical one-to-one mapping φ between the set $\mathscr{A}$ of numerical sequences and the set $\mathscr{B}$ of generating series of given type. These results allow one to formulate the following statement [32].

E-principle of summation: *each pair of inverse linear transforms* (*for sequences, series, functions, etc.*), *independently of the way of definition of the one-to-one mapping* φ, *generates the corresponding method of summation* (*the method of coefficients*).

Georgy P. Egorychev
Siberian Federal University, Krasnoyarsk, RUSSIA, e-mail: anott@scn.ru

I.S. Kotsireas, E.V. Zima (eds.), *Advances in Combinatorial Mathematics*,
DOI 10.1007/978-3-642-03562-3_1, © Springer-Verlag Berlin Heidelberg 2009

This principle provides for the first time a foundation for the classical method of generating functions (generating integrals) as a method of summation for different classes of generating series (the Completeness Lemma). It also makes it possible to reduce the variety of calculations with them to a uniform combinatorial scheme, and to set a new extensive program of solving open summation problems.

1.2 The method of generating functions as a method of summation (the method of coefficients)

1.2.1 Computational scheme

The general scheme of the method of integral representations of sums can be broken down into the following steps [25].

1. *Assignment of a table of integral representations of combinatorial numbers. For example, the binomial coefficients* $\binom{n}{k}$, *n*, $k = 0, 1, \ldots,$

$$\binom{n}{k} = \mathbf{res}_w (1+w)^n w^{-k-1} = \frac{1}{2\pi i}\int_{|w|=\rho} (1+w)^n w^{-k-1} dw,\ \rho > 0; \tag{1.1}$$

$$\binom{n+k-1}{k} = \mathbf{res}_w (1-w)^{-n} w^{-k-1} = \frac{1}{2\pi i}\int_{|w|=\rho} (1-w)^{-n} w^{-k-1} dw,\ 0 < \rho < 1. \tag{1.2}$$

Stirling numbers of the second kind $S_2(n,k)$, *n*, $k = 0, 1, \ldots$([25], p. 273): $S_2(0,0) := 1$, *and*

$$S_2(n,k) = \mathbf{res}_w\{(-1+\exp w)^n w^{-k-1}\} = \frac{1}{2\pi i}\int_{|w|=\rho} (-1+\exp w)^n w^{-k-1} dw,\ \rho > 0. \tag{1.3}$$

The Kronecker symbol $\delta(n,k)$, *n*, $k = 0, 1, \ldots,$

$$\delta(n,k) = \mathbf{res}_w w^{-n+k-1}. \tag{1.4}$$

2. *Representation of the summand* a_k *of the original sum* $\sum_k a_k$ *by a sum of product of combinatorial numbers.*

3. *Replacement of the combinatorial numbers by their integrals.*

4. *Reduction of products of integrals to multiple integral.*

5. *Interchange of the order of summation and integration. This gives the integral representation of original sum with the kernel represented by a series. The use of this transformation requires us to deform the domain of integration in such a way as to obtain the series under the integral which converges uniformly on this domain saving the value of the integral.*

6. *Summation of the series under the integral sign. As a rule, this series turns out to be a geometric progression* [46]. *This gives the integral representation of the original sum with the kernel in closed form.*

7. *Computation of the resulting integral by means of tables of integrals, iterated integration, the theory of one-dimensional and multidimensional residues, or other suitable methods.*

1.2.2 Operations with formal power series and the inference rules

Hans Rademacher [87] has noted, that the applications of the method of generating functions is connected usually with use of operations over the Laurent series and the Dirichlet series. Earlier the author has developed the method of integral representations and calculation of combinatorial sums of various types [25, 26, 29, 31], connected with use of the theory of analytic functions, the theory of multiple residues in $\mathbb{C}^n$ and the formal power Laurent series over $\mathbb{C}$. In this section we give an analogous construction and the foundation of the method of coefficients for classic formal Dirichlet series of one variable over $\mathbb{C}$.

1.2.2.1 Laurent power series: definition and properties of the residue operator

Using the **res** concept and its properties the idea of integral representations can be extended on sums that allow computation with the help of formal Laurent power series of one and several variables over $\mathbb{C}$. The **res** concept is directly connected with the classic concept of residue in the theory of analytic functions and which may be used with series of various types. This connection has enabled us to express properties of **res** operator analogous to properties of residue in the theory of analytic functions. This in turn allows us to unify the scheme of the method of integral representations independently of what kind of series – convergent or formal – is being used (separately, or jointly) in the process of computation of a particular sum.

In this section we shall restrict ourselves to explaining only one-dimensional case, although in further computations the **res** concept shall also be used for multivariate series. Besides, the one-dimensional case is interesting by itself in the computation of multiple integrals in terms of repeated integrals.

Let L be the set of formal Laurent power series over $\mathbb{C}$ containing only finitely many terms with negative degrees. The *order* of the monomial $c_k w^k$ is k. The *order* of the series $C(w) = \sum_k c_k w^k$ from L is the minimal order of monomials with nonzero coefficient. Let L_k denote the set of series of order k, $L = U_{k=-\infty}^{\infty} L_k$. Two series $A(w) = \sum_k a_k w^k$ and $B(w) = \sum_k b_k w^k$ from L are equal iff $a_k = b_k$ for all k. We can introduce in L operations of addition, multiplication, substitution, inversion and differentiation [15, 35, 47]. The ring L is a field [85]. Let $f(w)$, $\psi(w) \in L_0$. Below we shall use the following notations: $h(w) = wf(w) \in L_1$, $l(w) = w/\psi(w) \in L_1$, $z'(w) = \frac{d}{dw}z(w)$, $\overline{h} = \overline{h}(z) \in L_1$ – the inverse series of the series $z = h(w) \in L_1$.

For $C(w) \in L$ define the *formal residue* as

$$\mathbf{res}_w C(w) = c_{-1}. \tag{1.5}$$

Let $A(w) = \sum_k a_k w^k$ be the *generating function* for the sequence $\{a_k\}$. Then

$$a_k = \mathbf{res}_w A(w) w^{-k-1},\ k = 0, 1, ... \tag{1.6}$$

For example, one of the possible representations of the binomial coefficient is

$$\binom{n}{k} = \mathbf{res}_w\, (1+w)^n\, w^{-k-1},\ k = 0, 1, ..., n. \tag{1.7}$$

There are several properties (*inference rules*) for the **res** operator which immediately follow from its definition and properties of operations in formal Laurent power series over $\mathbb{C}$. We list only a few of them which will be used in this article. Let $A(w) = \sum_k a_k w^k$ and $B(w) = \sum_k b_k w^k$ be generating functions from L.
Rule 1 (Removal).

$$\mathbf{res}_w A(w) w^{-k-1} = \ \mathbf{res}_w B(w) w^{-k-1} \text{ for all } k \text{ iff } A(w) = B(w). \tag{1.8}$$

Rule 2 (Linearity). *For any* α, β *from* $\mathbb{C}$

$$\alpha\ \mathbf{res}_w A(w) w^{-k-1} + \beta\ \mathbf{res}_w B(w) w^{-k-1} = \ \mathbf{res}_w((\alpha A(w) + \beta B(w)) w^{-k-1}). \tag{1.9}$$

By induction from (1.9) *it follows, that the operators* $\sum$ *and* **res** *are commutative.*
Rule 3 (Substitution). a) *For* $w \in L_k$ $(k \geq 1)$ *and* $A(w)$ *any element of* L, *or* b) *for* $A(w)$ *polynomial and* w *any element of* L *including a constant*

$$\sum_k w^k \mathbf{res}_z \left(A(z) z^{-k-1}\right) = [A(z)]_{z=w} = A(w). \tag{1.10}$$

Rule 4 (Inversion). *For* $f(w)$ *from* L_0

$$\sum_k z^k \mathbf{res}_w \left(A(w) f(w)^k w^{-k-1}\right) = \left[A(w)/f(w)h'(w)\right]_{w=\overline{h(z)}}, \tag{1.11}$$

where $z = h(w) = wf(w) \in L_1$.
Rule 5 (Change of variables). *If* $f(w) \in L_0$, *then*

$$\mathbf{res}_w \left(A(w) f(w)^k w^{-k-1}\right) = \ \mathbf{res}_z(\left[A(w)/f(w)h'(w)\right]_{w=\overline{h(z)}} z^{-k-1}), \tag{1.12}$$

where $z = h(w) = wf(w) \in L_1$.
Rule 6 (Differentiation).

$$k\ \mathbf{res}_w A(w) w^{-k-1} = \ \mathbf{res}_w A'^{-k}. \tag{1.13}$$

1.2.2.2 Dirichlet series: definitions and properties of the $[q^{-s}]$ operator

Let H be the set of formal Dirichlet series $A(s)=\sum_{k\geq 1}a_k k^{-s}$ of usual type in formal variable s with complex coefficients. Two series $A(s)=\sum_k a_k k^{-s}$ and $B(s)=\sum_k b_k k^{-s}$ from H are equal iff $a_k=b_k$ for all k. We can introduce in H operations of addition, multiplication and differentiation of series [63, 70]. The set H is a ring.

Let G be the set of formal exponential series of type $A(s)=\sum_{q\in\mathbb{Q}}a_q q^{-s}$ in variable s with complex coefficients, $H\subset G$. For $A(s)\in G$ define *the* $[q^{-s}]$*-operator* as

$$a_q=\left[q^{-s}\right](A(s)),\ \forall q\in\mathbb{Q}, \tag{1.14}$$

i.e. the $[q^{-s}]$-operator is the coefficient at the exponent q^{-s} of the series $A(s)$. If $A(s)=\sum_k a_k k^{-s}$ from H is the generating function for the sequence $\{a_k\}$ then as usual

$$a_k=\left[k^{-s}\right](A(s)),\ k=1,2,\dots \tag{1.15}$$

Remark. Here the sign $\sum_{q\in\mathbb{Q}}$ is analogous to the sign $\sum_{k\in\mathbb{N}}$ which we often use instead the sign $\sum_{k=0}^{\infty}$ for power series and formal Dirichlet series of usual type (see also [85], p.118). The notion of the formal exponential series $A(s)=\sum_{q\in\mathbb{Q}}a_q q^{-s}$ from G is necessary below in the proof of formulae in section 1.2.3.

For example, we have the following representation for the coefficients of zeta-function $\zeta(s):=\sum_{k\geq 1}k^{-s}$, and the inverse of it $1/\zeta(s)=\sum_{k\geq 1}\mu(k)k^{-s}$, Re $s>-1$:

$$1=\left[k^{-s}\right](\zeta(s)),\ k=1,2,\dots, \tag{1.16}$$

$$\mu(k)=\left[k^{-s}\right](1/\zeta(s)),\ k=1,2,\dots, \tag{1.17}$$

where μ is the Möbius function.

There are several properties (*inference rules*) for the $[q^{-s}]$-operator which immediately follow from its definition and properties of operations on the formal Dirichlet series over $\mathbb{C}$. Let $A(s)=\sum_k a_k k^{-s}$ and $B(s)=\sum_k b_k k^{-s}$ be the generating functions for the sequences $\{a_k\}$ and $\{b_k\}$from H.

Rule 1 (Removal).

$$\left[k^{-s}\right](A(s))=\left[k^{-s}\right](B(s))\text{ for all }k\text{ iff }A(s)=B(s). \tag{1.18}$$

Rule 2 (Shifting). For any $d,n\in\mathbb{N}$

$$[(n/d)^{-s}](A(s))=[n^{-s}](d^{-s}A(s)). \tag{1.19}$$

Rule 3 (Linearity). *For any* α,β *from* $\mathbb{C}$

$$\alpha\left[q^{-s}\right](A(s))+\beta\left[q^{-s}\right](B(s))=\left[q^{-s}\right](\alpha A(s)+\beta B(s)). \tag{1.20}$$

By induction from (1.20) *follows, that operators* $\sum$ *and* $[q^{-s}]$ *commute.*

Rule 4 (Substitution).

$$\sum_{k\geq 1}k^{-s}\left[k^{-t}\right](A(t))=(A(t))|_{t=s}=A(s). \tag{1.21}$$

Rule 5 (Differentiation).

$$\left[k^{-s}\right](A'(s)) = -\ln k \times \left[k^{-s}\right](A(s)), k = 1,2,.... \tag{1.22}$$

1.2.3 The problem of completeness

1.2.3.1 Statement of the problem

In solving analytic problems with the help of generating functions we usually encounter one of the following interconnected problems.

Problem A. *Suppose that a series $S(w) = \sum_k s_k w^k$ from L is expressed in terms of the series $A(w) = \sum_k a_k w^k$, $B(w) = \sum_k b_k w^k$,..., $D(w) = \sum_k d_k w^k$ from L with the help of different operations on the formal Laurent power series over $\mathbb{C}$, i.e. the formula*

$$S(w) = F(A(w), B(w), \dots, D(w)) \tag{1.23}$$

is given. For each k find the formula

$$s_k = f(\{a_k\}, \{b_k\}, \dots, \{d_k\}) \tag{1.24}$$

for the terms of sequence $\{s_k\}$ as a function of the terms of sequences $\{a_k\}$, $\{b_k\}, \dots, \{d_k\}$.

Definition. *A sequence $\{s_k\}$ is called of A-type with respect to terms of sequences $\{a_k\}, \{b_k\}, \dots, \{d_k\}$, if it is determined by a formula of type (1.24).*

Problem B. *Let for each k the formula $s_k = f(\{a_k\}, \{b_k\}, \dots, \{d_k\})$, $\forall k = 0, 1, \cdots$, with respect to terms of number sequences $\{a_k\}, \{b_k\}, \dots, \{d_k\}$ be given, but a functional dependence (1.23) between its generating functions is unknown. It is required to find out, whether the initial formula $s_k = f(\{a_k\}, \{b_k\}, \dots, \{d_k\})$ is a formula of A-type, and if yes, then to find formula $S(w) = F(A(w), B(w), \dots, D(w))$.*

Definition. *A set of rules for* **res** *operator ($[q^{-s}]$-operator) is called complete, if it allows one to solve problem* ***B***.

1.2.3.2 Completeness Lemma: Laurent and Dirichlet series

Completeness Lemma.

(a) *The set of rules 1 – 6 for the* **res** *operator of the formal Laurent series is complete* [26].

(b) *The set of rules 1 – 5 for the $[q^{-s}]$-operator of the formal Dirichlet series of usual type is complete.*

Proof.

(a) In [25] (pp. 31–35) and [26] we use induction on the number of different operations over sequences $\{a_k\}$, $\{b_k\}$, ..., $\{d_k\}$ in (1.24) generating the given sequence $\{s_k\}$. On the first step of induction a series $S(w)$ is obtained with the help of series $A(w)$ and $B(w)$ from L by one operation over formal Laurent power series (addition, multiplication, etc.).

(b) Below we perform analogous calculations for the formal Dirichlet series of usual type. On the first step of induction a series $S(s)$ is obtained with the help of the formal Dirichlet series $A(s)$ and $B(s)$ from H and one of the operations of addition and multiplication. We should give the solution to recursive relations that corresponds to each of these operations.

Addition operation. If $c_k = a_k + b_k$, $k = 1,2,\ldots$, then by formulae (1.15) for the coefficients c_k, a_k and b_k we obtain

$$\left[k^{-s}\right](C(s)) = \left[k^{-s}\right](A(s)) + \left[k^{-s}\right](B(s)),\ k = 1,2,\ldots,$$

(by the linearity rule and the removal rule)

$$\Leftrightarrow \left[k^{-s}\right](C(s)) = \left[k^{-s}\right](A(s) + (B(s)) \text{ for all } k \ \Leftrightarrow C(s) = A(s) + B(s).$$

Multiplication operation. On one hand we have $C(s) = A(s) \times B(s) := \sum_k c_k k^{-s}$, where

$$c_k = \sum_{d|k} a_d b_{k/d},\ k = 1,2,\ldots, \tag{1.25}$$

where (and up to the end of the section) the summation is over all the divisors d of natural number k. Conversely, if the identity (1.25) holds, then for $k = 1,2,\ldots$, we get:

$$c_k = \sum_{d|k} a_d b_{k/d},$$

(the change of coefficients a_d and $b_{k/d}$ by formulae (1.15))

$$\sum_{d|k} \left[d^{-t}\right](A(t)) \times \left[(k/d)^{-s}\right](B(s)) = \sum_{d=1}^{\infty} \left[d^{-t}\right](A(t)) \times \left[(k/d)^{-s}\right](B(s))$$

(as added terms are equal to zero by the definition (1.14) of the $[q^{-s}]$-operator, and further the shifting rule over s)

$$= \sum_{d=1}^{\infty} \left[d^{-t}\right]\left\{\left[k^{-s}\right](d^{-s}A(t)B(s))\right\}\ldots$$

(interchanging the order of $\sum$ and $\left[d^{-t}\right]\left[k^{-s}\right]$ and splitting the sum over the index d)

$$= \left[k^{-s}\right]\left(B(s) \times \left\{\sum_{d=1}^{\infty} d^{-s}\left[d^{-t}\right](A(t))\right\}\right)$$

(the substitution rule for an expression in braces and the change $t = s$)

$$= \left[k^{-s}\right]\left\{B(s) \times (A(t))|_{t=s}\right\} = \left[k^{-s}\right]\left\{B(s)A(s)\right\}.$$

Now by (1.25) we have

$$c_k := \left[k^{-s}\right](C(s)) = \left[k^{-s}\right]\{B(s)A(s)\},\ k = 1,2,...,$$

and the removal rule of the $[k^{-s}]$-operator gives us the required formula

$$C(s) = B(s)A(s).$$

If the hypothesis of Lemma holds for $n-1$ operations, then the next inductive step is similar to the initial step.

In the following illustrative example we use only concepts and the inference rules for the formal Dirichlet series.

Example. The celebrated *Möbius inversion formula* states that

$$f(n) = \sum_{d|n} g(d),\ n = 1,2,\ldots \Leftrightarrow g(n) = \sum_{d|n} \mu(d) f(n/d),\ n = 1,2,\ldots. \quad (1.26)$$

Proof. Let $F(s) = \sum_{n\geq 1} f(n) n^{-s}$ and $G(s) = \sum_{n\geq 1} g(n) n^{-s}$ from H be the generating functions for the sequences $\{f(n)\}$ and $\{g(n)\}$. Repeating the same scheme of calculations we get:

$$g(n) := [n^{-s}](G(s)) = \sum_{d|n} \mu(d) f(n/d)$$

(the substitution using (1.17) and (1.15): $f(n/d) = [(n/d)^{-s}](F(s))$ and $\mu(d) = [d^{-t}](1/\zeta(t))$

$$= \sum_{d|n} [d^{-t}](1/\zeta(t)) \times [(n/d)^{-s}](F(s)) = \sum_{d|n} \left[d^{-t}\right](1/\zeta(t)) \times \left[n^{-s}\right](d^{-s} F(s))$$

$$= \sum_{d\geq 1} \ldots = \left[n^{-s}\right]\left(F(s) \times \left\{\sum_{d=1}^{\infty} d^{-s}\left[d^{-t}\right](1/\zeta(t))\right\}\right)$$

$$= \left[n^{-s}\right](F(s) \times (1/\zeta(t))|_{t=s}) = \left[n^{-s}\right](F(s)/\zeta(s)).$$

Thus we obtain

$$[n^{-s}](G(s)) = \left[n^{-s}\right](F(s)/\zeta(s)),\ \text{for all}\ n,$$

and the removal rule of the $[n^{-s}]$-operator gives us

$$G(s) = F(s)/\zeta(s) \Leftrightarrow F(s) = \zeta(s) G(s) = \sum_{k\geq 1} k^{-s} \times \sum_{k\geq 1} g_k k^{-s},$$

i.e.

$$f(n) = \sum_{d|n} g(d),\ n = 1,2,\ldots.$$

Remark. *Completeness Lemma supports the possibility of finding with the help of the method of coefficients an operational (integral) representation for those sums, which admit the calculation with formal Laurent power series and Dirichlet formal series with complex coefficients. Basic difficulty in the use of this method (the set of inference rules and the Completeness Lemma) consists in the solution of problems of classification and recognition of expressions of A-type, and in construction of algorithms of induction search though these problems have found the successful solution in many concrete cases of calculation of combinatorial sums* [25].

1.2.4 Connection with the theory of analytic functions

If a formal power series $A(w) \in L$ converges in a punctured neighborhood of zero, then the definition of $\mathbf{res}_w A(w)$ coincides with the usual definition of $\mathbf{res}_{w=0} A(w)$, used in the theory of analytic functions. The formula (1.6) is an analog of the well-known integral Cauchy formula

$$a_k = \frac{1}{2\pi i} \oint_{|w|=\rho} A(w) w^{-k-1} dw$$

for the coefficients of the Taylor series in a punctured neighborhood of zero. The substitution rule (1.10) of the **res** operator is a direct analog of the famous Cauchy theorem. Similarly, it is possible to introduce the definition of formal residue at the point of infinity, the logarithmic residue and the theorem of residues (all necessary concepts and results in the theory of residues in one and several complex variables, see [2, 25, 34, 76, 95, 107]). Moreover, it is easy to see that each rule of the **res** operator can be simply proven by reduction to the known formula in the theory of residues for corresponding rational function [25].

The theory of Dirichlet series of usual type can be found in many books on the theory of holomorphic functions and analytical number theory (see, for example, [63, 70]).

1.3 Several recent applications

1.3.1 The characteristic function of the stopping height for the Collatz conjecture

The $3x+1$ problem is known under different names. It is often called *Collatz problem, Ulam problem, the Syracuse problem, Kakutani problem, and Hasse algorithm* [60]. Consider the sequence of iterations $(n, f(n), f(f(n)), \ldots)$, where

$$f(n)=\begin{cases}(3n+1)/2, & \text{for odd } n,\\ n/2, & \text{for even } n.\end{cases} \tag{1.27}$$

The $3x+1$ conjecture states that for any natural number n this sequence will contain the number 1. The index of the first element equal to 1 in this sequence is called *stopping height* of the instance of Collatz problem and is denoted $\sigma(n)$.

The following arithmetic reformulation of the Collatz problem is given in [71].

Theorem 1.[1] *The $3x+1$ conjecture is true iff for every positive integer a there are natural numbers w and v such that $a\le w$ and*

$$\binom{2w+1}{w}\binom{4(w+1)v+1}{v}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\binom{v}{r}\binom{w(v-r)}{s}\binom{wr}{t}\times \tag{1.28}$$

$$\binom{2s+2t+r+(4w+3)v+1}{3((4w+4)t+a)+2(4w+4)r+(4w+4)s}\times$$

$$\binom{3((4w+4)t+a)+2(4w+4)r+(4w+4)s}{2s+2t+r+(4w+3)v+1}\equiv 1 \pmod 2.$$

In [27, 31] one can find the following reformulation of (1.28) obtained with the help of the method of coefficients and based on congruences (modulo 2)

$$(1+u)^{\alpha}\equiv 1+u^{\alpha},\ \ (1+u)^{\alpha-1}\equiv\sum_{s=0}^{\alpha-1}u^{s},\ \ (1-(\alpha-1)^2u)^{-1/(\alpha-1)}\equiv\prod_{s=0}^{\infty}\left(1+u^{\alpha^{s}}\right), \tag{1.29}$$

where $\alpha=2^{x}$, $x\in\mathbb{N}$:

Let $a,v,w\in\mathbb{N}$ and denote

$$S=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\binom{v}{r}\binom{w(v-r)}{s}\binom{wr}{t}\binom{2s+2t+r+(4w+3)v+1}{3(4w+4)t+2(4w+4)r+(4w+4)s+a}\times$$

$$\binom{3(4w+4)t+2(4w+4)r+(4w+4)s+a}{2s+2t+r+(4w+3)v+1}. \tag{1.30}$$

Then

$$S=\mathbf{res}_u\{g(u)\,u^{-(4w+3)v+a-2}\}, \tag{1.31}$$

where

$$g(u)=\left(\left(1+u^{-2+(4w+4)}\right)^{w}+u^{-1+2(4w+4)}\left(1+u^{-2+3(4w+4)}\right)^{w}\right)^{v}. \tag{1.32}$$

This leads to the following reformulation of $3x+1$ conjecture.

Theorem 2 [27]. *The $3x+1$ conjecture is true iff for every positive integer a there are natural numbers r and $\alpha=2^{x+2}$, where $x\in\mathbb{N}$, such that $a\le -1+\alpha/4$,*

[1] Careful investigation of this result along with computer experiments shows that this formula and analogous statements ([71], Theorem 1, Corollary 1 – 3) are not valid. The following correction is required: the term a has to be replaced by $a/3$ in order to make it work. We shall use the corrected version of (1.28) below.

and the following congruence is true

$$\mathbf{res}_u u^{-\alpha^r+a-1}\prod_{t=0}^{\infty}\left(\sum_{s=0}^{-1+\alpha/4}\left(u^{s(-2+\alpha)\alpha^t}+u^{(-1+2\alpha+s(-2+3\alpha))\alpha^t}\right)\right)\equiv 1 \pmod 2, \tag{1.33}$$

1.3.1.1 Properties of the characteristic function of the stopping height

Definition. *In accordance with* (1.33) *denote*

$$Q_\alpha(u) := d_\alpha(u)\prod_{t=1}^{\infty} d_\alpha(u^{\alpha^t}), \tag{1.34}$$

where the polynomial

$$d_\alpha(u) = 1 + u^{-1+2\alpha} + \sum_{s=1}^{-1+\alpha/4}\left(u^{s(-2+\alpha)} + u^{-1+2\alpha+s(-2+3\alpha)}\right).$$

It is shown in [27], that the coefficients of this formal power series $Q_\alpha(u)$ over integers

$$Q_\alpha(u) = \sum_k q_k(\alpha)u^k \tag{1.35}$$

are equal to either 0 or 1. Therefore, the congruence (1.33) is a theoretical-functional reformulation of the Collatz conjecture. It was noted in [27, 28], that the parameter r in Theorem 2 is equal to the stopping height $\sigma(n)$. Thus under the assumptions of Theorem 2, now the equivalent formulation of the Collatz conjecture can be given by the equality

$$q_{-n+\alpha^{\sigma(n)}} = 1. \tag{1.36}$$

The last formulation is more attractive than (1.28), and these properties of the function $Q_\alpha(u)$ allows us to call it *the characteristic function of the stopping height in the Collatz conjecture*.

Lemma (*Characteristic property*) *For any* α *the coefficients of the formal power series* $Q_\alpha(u) \in H(\mathbb{Z})$ *in* (1.35) *are equal to either* 0 or 1.

Proof. The statement of Lemma was proven in [27] only for $k = \alpha^q - n,\ n \in \mathbb{N}$. However, that proof can be repeated for an arbitrary k.

Lemma (*Functional equations*) *For any* α, *the function* $Q_\alpha(u)$ *is uniquely defined by the functional equation*

$$Q_\alpha(0) = 1, \quad Q_\alpha(u) = d_\alpha(u)Q_\alpha(u^\alpha). \tag{1.37}$$

The following congruence holds

$$(d_\alpha(u))^{-1/(\alpha-1)} \equiv Q_\alpha(u) (\text{mod } 2), \tag{1.38}$$