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The Comprehensive Dictionary of Mathematics
Stan Gibilisco

A VOLUME IN THE
COMPREHENSIVE DICTIONARY
OF MATHEMATICS

DICTIONARY OF ANALYSIS, CALCULUS, AND DIFFERENTIAL EQUATIONS

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CRC Press
Boca Raton London New York Washington, D.C.

Library of Congress Cataloging-in-Publication Data

Dictionary of analysis, calculus, and differential equations / [edited by] Douglas N. Clark.
p. cm. — (Comprehensive dictionary of mathematics)
ISBN 0-8493-0320-6 (alk. paper)
1. Mathematical analysis—Dictionaries. 2. Calculus—Dictionaries. 3. Differential equations—Dictionaries. I. Clark, Douglas N. (Douglas Napier), 1944– II. Series.

QA5 .D53 1999
515′.03—dc21 99-087759

Direct all inquiries to CRC Press LLC, 2000 N.W. Corporate Blvd., Boca Raton, Florida 33431.

International Standard Book Number 0-8493-0320-6
Library of Congress Card Number 99-087759
Printed in the United States of America 2 3 4 5 6 7 8 9 0
Printed on acid-free paper

Preface

Book 1 of the CRC **Press Comprehensive Dictionary of Mathematics** covers analysis, calculus, and differential equations broadly, with overlap into differential geometry, algebraic geometry, topology, and other related fields. The authorship is by 15 mathematicians, active in teaching and research, including the editor.

Because it is a dictionary and not an encyclopedia, definitions are only occasionally accompanied by a discussion or example. Because it is a dictionary of mathematics, the primary goal has been to define each term rigorously. The derivation of a term is almost never attempted.

The dictionary is written to be a useful reference for a readership which includes students, scientists, and engineers with a wide range of backgrounds, as well as specialists in areas of analysis and differential equations and mathematicians in related fields. Therefore, the definitions are intended to be accessible, as well as rigorous. To be sure, the degree of accessibility may depend upon the individual term, in a dictionary with terms ranging from Albanese variety to z intercept.

Occasionally a term must be omitted because it is archaic. Care was takenwhen such circumstances arose because an archaic term may not be obsolete. An example of an archaic term deemed to be obsolete, and hence not included, is right line. This term was used throughout a turn-of-the-century analytic geometry textbook we needed to consult, but it was not defined there. Finally, reference to a contemporary English language dictionary yielded straight line as a synonym for right line.

The authors are grateful to the series editor, Stanley Gibilisco, for dealing with our seemingly endless procedural questions and to Nora Konopka, for always acting efficiently and cheerfully with CRC Press liaison matters.

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A

a.e. *See* almost everywhere.

Abel summability A series $\sum_{j=0}^{\infty} a_j$ is *Abel summable* to A if the power series

$$f(z) = \sum_{j=0}^{\infty} a_j z^j$$

converges for $|z| < 1$ and

$$\lim_{x \to 1-0} f(x) = A.$$

A consequence is that convergence of the series $\sum a_j$ to the limit L implies *Abel summability* of the series to L.

Abel's Continuity Theorem *See* Abel's Theorem.

Abel's integral equation The equation

$$\int_a^x \frac{u(t)}{(x-t)^\alpha} dt = f(x),$$

where $0 < \alpha < 1$, $a \leq x \leq b$ and the given function $f(x)$ is $\mathbf{C}^1$ with $f(a) = 0$. A continuous solution $u(x)$ is sought.

Abel's problem A wire is bent into a planar curve and a bead of mass m slides down the wire from initial point (x, y). Let $T(y)$ denote the time of descent, as a function of the initial height y. *Abel's mechanical problem* is to determine the shape of the wire, given $T(y)$. The problem leads to *Abel's integral equation:*

$$\frac{1}{\sqrt{2g}} \int_0^y \frac{f(v)}{\sqrt{y-v}} dv = T(y).$$

The special case where $T(y)$ is constant leads to the *tautochrone.*

Abel's Theorem Suppose the power series $\sum_{j=0}^{\infty} a_j x^j$ has radius of convergence R and that $\sum_{j=0}^{\infty} a_j R^j < \infty$, then the original series converges uniformly on $[0, R]$.

Abelian differential An assignment of a meromorphic function f to each local coordinate z on a Riemann surface, such that $f(z)\mathrm{d}z$ is invariantly defined. Also *meromorphic differential.*

Sometimes, analytic differentials are called *Abelian differentials of the first kind,* meromorphic differentials with only singularities of order ≥ 2 are called *Abelian differentials of the second kind,* and the term *Abelian differential of the third kind* is used for all other Abelian differentials.

Abelian function An inverse function of an Abelian integral. *Abelian functions* have two variables and four periods. They are a generalization of elliptic functions, and are also called *hyperelliptic functions. See also* Abelian integral, elliptic function.

Abelian integral (**1.**) An integral of the form

$$\int_0^x \frac{dt}{\sqrt{P(t)}},$$

where $P(t)$ is a polynomial of degree > 4. They are also called hyperelliptic integrals.

See also Abelian function, elliptic integral of the first kind.

(**2.**) An integral of the form $\int R(x, y)dx$, where $R(x, y)$ is a rational function and where y is one of the roots of the equation $F(x, y) = 0$, of an algebraic curve.

Abelian theorems Any theorems stating that convergence of a series or integral implies summability, with respect to some summability method. *See* Abel's Theorem, for example.

abscissa The first or x-coordinate, when a point in the plane is written in rectangular coordinates. The second or y-coordinate is called the *ordinate.* Thus, for the point (x, y), x is the abscissa and y is the ordinate. The *abscissa* is the horizontal distance of a point from the y-axis and the ordinate is the vertical distance from the x-axis.

abscissa of absolute convergence The unique real number σ_a such that the Dirichlet series

$$\sum_{j=1}^{\infty} a_j e^{-\lambda_j s}$$

(where $0 < \lambda_1 < \lambda_2 \cdots \to \infty$) converges absolutely for $\Re s > \sigma_a$, and fails to converge absolutely for $\Re s < \sigma_a$. If the Dirichlet series converges for all s, then the abscissa of absolute convergence $\sigma_a = -\infty$ and if the Dirichlet series never converges absolutely, $\sigma_a = \infty$. The vertical line $\Re s = \sigma_a$ is called the *axis of absolute convergence.*

abscissa of boundedness The unique real number σ_b such that the sum $f(s)$ of the Dirichlet series

$$f(s) = \sum_{j=1}^{\infty} a_j e^{-\lambda_j s}$$

(where $0 < \lambda_1 < \lambda_2 \cdots \to \infty$) is bounded for $\Re s \geq \sigma_b + \delta$ but not for $\Re s \geq \sigma_b - \delta$, for every $\delta > 0$.

abscissa of convergence (**1.**) The unique real number σ_c such that the Dirichlet series

$$\sum_{j=1}^{\infty} a_j e^{-\lambda_j s}$$

(where $0 < \lambda_1 < \lambda_2 \cdots \to \infty$) converges for $\Re s > \sigma_c$ and diverges for $\Re s < \sigma_c$. If the Dirichlet series converges for all s, then the abscissa of convergence $\sigma_c = -\infty$, and if the Dirichlet series never converges, $\sigma_c = \infty$. The vertical line $\Re s = \sigma_c$ is called the *axis of convergence.*
(**2.**) A number σ such that the Laplace transform of a measure converges for $\Re z > \sigma$ and does not converge in $\Re z > \sigma - \epsilon$, for any $\epsilon > 0$. The line $\Re z = \sigma$ is called the *axis of convergence.*

abscissa of regularity The greatest lower bound σ_r of the real numbers σ' such that the function $f(s)$ represented by the Dirichlet series

$$f(s) = \sum_{j=1}^{\infty} a_j e^{-\lambda_j s}$$

(where $0 < \lambda_1 < \lambda_2 \cdots \to \infty$) is regular in the half plane $\Re s > \sigma'$. Also called *abscissa of holomorphy.* The vertical line $\Re s = \sigma_r$ is called the *axis of regularity.* It is possible that the abscissa of regularity is actually less than the abscissa of convergence. This is true, for example, for the Dirichlet series $\sum(-1)^j j^{-s}$, which converges only for $\Re s > 0$; but the corresponding function $f(s)$ is entire.

abscissa of uniform convergence The unique real number σ_u such that the Dirichlet series

$$\sum_{j=1}^{\infty} a_j e^{-\lambda_j s}$$

(where $0 < \lambda_1 < \lambda_2 \cdots \to \infty$) converges uniformly for $\Re s \geq \sigma_u + \delta$ but not for $\Re s \geq \sigma_u - \delta$, for every $\delta > 0$.

absolute continuity (**1.**) For a real valued function $f(x)$ on an interval $[a, b]$, the property that, for every $\epsilon > 0$, there is a $\delta > 0$ such that, if $\{(a_j, b_j)\}$ are intervals contained in $[a, b]$, with $\sum(b_j - a_j) < \delta$ then $\sum |f(b_j) - f(a_j)| < \epsilon$.
(**2.**) For two measures μ and ν, absolute continuity of μ with respect to ν (written $\mu << \nu$) means that whenever E is a ν-measurable set with $\nu(E) = 0$, E is μ-measurable and $\mu(E) = 0$.

absolute continuity in the restricted sense Let $E \subset \mathbf{R}$, let $F(x)$ be a real-valued function whose domain contains E. We say that F is *absolutely continuous in the restricted sense* on E if, for every $\epsilon > 0$ there is a $\delta > 0$ such that for every sequence $\{[a_n, b_n]\}$ of non-overlapping intervals whose endpoints belong to E, $\sum_n (b_n - a_n) < \delta$ implies that $\sum_n O\{F; [a_n, b_n]\} < \epsilon$. Here, $O\{F; [a_n, b_n]\}$ denotes the oscillation of the function F in $[a_n, b_n]$, i.e., the difference between the least upper bound and the greatest lower bound of the values assumed by $F(x)$ on $[a_n, b_n]$.

absolute convergence (**1.**) For an infinite series $\sum_{n=1}^{\infty} a_j$, the finiteness of $\sum_{j=1}^{\infty} |a_j|$. (**2.**) For an integral

$$\int_S f(x)dx,$$

the finiteness of

$$\int_S |f(x)|dx.$$

absolute curvature The absolute value

$$|k| = \left|\frac{d^2r}{ds^2}\right| = +\sqrt{\left|g_{ik}\frac{D}{ds}\left(\frac{dx^i}{ds}\right)\frac{D}{ds}\left(\frac{dx^k}{ds}\right)\right|}$$

of the first curvature vector $\frac{d^2r}{ds^2}$ is the *absolute curvature* (*first,* or *absolute geodesic curvature*) of the regular arc C described by n parametric equations

$$x^i = x^i(t) \quad (t_1 \le t \le t_2)$$

at the point $(x^1, x^2, \dots, x^n)$.

absolute maximum A number M, in the image of a function $f(x)$ on a set S, such that $f(x) \le M$, for all $x \in S$.

absolute minimum A number m, in the image of a function $f(x)$ on a set S, such that $f(x) \ge m$ for all $x \in S$.

absolute value For a real number a, the absolute value is $|a| = a$, if $a \ge 0$ and $|a| = -a$ if $a < 0$. For a complex number $\zeta = a + bi$, $|\zeta| = \sqrt{a^2+b^2}$. Geometrically, it represents the distance from $0 \in \mathbf{C}$. Also called *amplitude, modulus.*

absolutely continuous spectrum *See* spectral theorem.

absolutely convex set A subset of a vector space over $\mathbf{R}$ or $\mathbf{C}$ that is both convex and balanced. *See* convex set, balanced set.

absolutely integrable function *See* absolute convergence (for integrals).

absorb For two subsets A, B of a topological vector space X, A is said to *absorb* B if, for some nonzero scalar α,

$$B \subset \alpha A = \{\alpha x : x \in A\}.$$

absorbing A subset M of a topological vector space X over $\mathbf{R}$ or $\mathbf{C}$, such that, for any $x \in X$, $\alpha x \in M$, for some $\alpha > 0$.

abstract Cauchy problem Given a closed unbounded operator T and a vector v in the domain of T, the *abstract Cauchy problem* is to find a function f mapping $[0, \infty)$ into the domain of T such that $f'(t) = Tf$ and $f(0) = v$.

abstract space A formal system defined in terms of geometric axioms. Objects in the space, such as lines and points, are left undefined. Examples include abstract vector spaces, Euclidean and non-Euclidean spaces, and topological spaces.

acceleration Let $p(t)$ denote the position of a particle in space, as a function of time. Let

$$s(t) = \int_0^t \left((\frac{dp}{dt}, \frac{dp}{dt})\right)^{\frac{1}{2}} dt$$

be the length of path from time $t = 0$ to t. The *speed* of the particle is

$$\frac{ds}{dt} = \left((\frac{dp}{dt}, \frac{dp}{dt})\right) = \|\frac{dp}{dt}\|,$$

the *velocity* $\mathbf{v}(t)$ is

$$\mathbf{v}(t) = \frac{dp}{dt} = \frac{dp}{ds}\frac{ds}{dt}$$

and the *acceleration* $\mathbf{a}(t)$ is

$$\mathbf{a}(t) = \frac{d^2p}{dt^2} = \frac{dT}{ds}\left(\frac{ds}{dt}\right)^2 + T\frac{d^2s}{dt^2},$$

where T is the unit tangent vector.

accretive operator A linear operator T on a domain D in a Hilbert space H such that $\Re(Tx, x) \geq 0$, for $x \in D$. By definition, T is accretive if and only if $-T$ is *dissipative.*

accumulation point Let S be a subset of a topological space X. A point $x \in X$ is an *accumulation point* of S if every neighborhood of x contains infinitely many points of $E \backslash \{x\}$.

Sometimes the definition is modified, by replacing "infinitely many points" by "a point."

addition formula A functional equation involving the sum of functions or variables. For example, the property of the exponential function:

$$e^a \cdot e^b = e^{a+b}.$$

additivity for contours If an arc γ is subdivided into finitely many subarcs, $\gamma = \gamma_1 + \ldots + \gamma_n$, then the contour integral of a function $f(z)$ over γ satisfies

$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \ldots + \int_{\gamma_n} f(z)dz.$$

adjoint differential equation Let

$$L = a_0 \frac{d^n}{dt^n} + a_1 \frac{d^{n-1}}{dt^{n-1}} + \ldots + a_n$$

be a differential operator, where $\{a_j\}$ are continuous functions. The adjoint differential operator is

$$L^+ = (-1)^n \left(\frac{d^n}{dt^n}\right) M_{\bar{a}_0} + (-1)^{n-1} \left(\frac{d^{n-1}}{dt^{n-1}}\right) M_{\bar{a}_1} + \ldots + M_{\bar{a}_n}$$

where M_g is the operator of multiplication by g. The *adjoint differential equation* of $Lf = 0$ is, therefore, $L^+ f = 0$.

For a system of differential equations, the functions $\{a_j\}$ are replaced by matrices of functions and each $\bar{a}_j$ above is replaced by the conjugate-transpose matrix.

adjoint operator For a linear operator T on a domain D in a Hilbert space H, the adjoint domain is the set $D^* \subset H$ of all $y \in H$ such that there exists $z \in H$ satisfying

$$(Tx, y) = (x, z),$$

for all $x \in D$. The *adjoint operator* T^* of T is the linear operator, with domain D^*, defined by $T^* y = z$, for $y \in D^*$, as above.

adjoint system *See* adjoint differential equation.

admissible Baire function A function belonging to the class on which a functional is to be minimized (in the calculus of variations).

AF algebra A $\mathbf{C}^*$ algebra $\mathcal{A}$ which has an increasing sequence $\{A_n\}$ of finite-dimensional $\mathbf{C}^*$ subalgebras, such that the union $\cup_n A_n$ is dense in $\mathcal{A}$.

affine arc length (**1.**) For a plane curve $\mathbf{x} = \mathbf{x}(t)$, with

$$(\frac{d\mathbf{x}}{dt}, \frac{d^2\mathbf{x}}{dt^2}) \neq 0,$$

the quantity

$$s = \int (\frac{d\mathbf{x}}{dt}, \frac{d^2\mathbf{x}}{dt^2}).$$

(**2.**) For a curve $\mathbf{x}(p) = \{x_1(p), x_2(p), x_3(p)\}$ in 3-dimensional affine space, the quantity

$$s = \int \det \begin{pmatrix} x_1 & x_2 & x_3 \\ x_1' & x_2' & x_3' \\ x_1'' & x_2'' & x_3'' \end{pmatrix}^{\frac{1}{6}} dt.$$

affine connection Let B be the bundle of frames on a differentiable manifold M of dimension n. An *affine connection* is a connection on B, that is, a choice $\{H_b\}_{b \in B}$, of

subspaces $H_b \subset B_b$, for every $b \in B$, such that
(i.) $B_b = H_b + V_b$ (direct sum) where V_b is the tangent space at b to the fiber through b;
(ii.) $H_{bg} = g_*(H_b)$, for $g \in \mathrm{GL}(n, \mathbf{R})$; and
(iii.) H_b depends differentiably on b.

affine coordinates Projective space P^n is the set of lines in $\mathbf{C}^{n+1}$ passing through the origin. *Affine coordinates* in P^n can be chosen in each patch $U_j = \{[(x_0, x_1, \ldots, x_n)] : x_j \neq 0\}$ (where $[(x_1, x_2, \ldots, x_n)]$ denotes the line through 0, containing the point $(x_0, x_1, \ldots, x_n)$). If $z = [(z_0, \ldots, z_n)]$, with $z_j \neq 0$, the affine coordinates of z are $(z_0/z_j, \ldots, z_{j-1}/z_j, z_{j+1}/z_j, \ldots, z_n/z_j)$. Also called *nonhomogeneous coordinates.*

affine curvature (**1.**) For a plane curve $\mathbf{x} = \mathbf{x}(t)$, the quantity

$$\kappa = (\mathbf{x}'', \mathbf{x}''')$$

where $' = \frac{d}{ds}$, (arc length derivative).
(**2.**) For a space curve $\mathbf{x}(p) = \{x_1(p), x_2(p), x_3(p)\}$, the quantity

$$\kappa = \det \left\{ \begin{matrix} x_1^{(4)} & x_2^{(4)} & x_3^{(4)} \\ x_1' & x_2' & x_3' \\ x_1''' & x_2''' & x_3''' \end{matrix} \right\},$$

where derivatives are with respect to affine arc length.

One also has the first and second affine curvatures, given by

$$\kappa_1 = -\frac{\kappa}{4}, \kappa_2 = \frac{\kappa'}{4} - \tau,$$

where τ is the affine torsion. *See* affine torsion.

affine diffeomorphism A diffeomorphism q of n-dimensional manifolds induces maps of their tangent spaces and, thereby, a $\mathrm{GL}(n, \mathbf{R})$-equivariant diffeomorphism of their frame bundles. If each frame bundle carries a connection and the induced map of frame bundles carries one connection to the other, then q is called an *affine diffeomorphism,* relative to the given connections.

affine differential geometry The study of properties invariant under the group of affine transformations. (The general linear group.)

affine length Let X be an affine space, V a singular metric vector space and k a field of characteristic different from 2. Then (X, V, k) is a metric affine space with metric defined as follows. If x and y are points in X, the unique vector A of V such that $Ax = y$ is denoted by $\overrightarrow{x, y}$. The square affine length (distance) between points x and y of X is the scalar $\overrightarrow{x, y}^{\,2}$.

If (X, V, R) is Euclidean space, $\overrightarrow{x, y}^{\,2} \geq 0$ and the Euclidean distance between the points x and y is the nonnegative square root $\sqrt{\overrightarrow{x, y}^{\,2}}$. In this case, the square distance is the square of the Euclidean distance. One always prefers to work with the distance itself rather than the square distance, but this is rarely possible. For instance, in the Lorentz plane $\overrightarrow{x, y}^{\,2}$ may be negative and, therefore, there is no real number whose square is $\overrightarrow{x, y}^{\,2}$.

affine minimal surface The extremal surface of the variational problem $\delta\Omega = 0$, where Ω is affine surface area. It is characterized by the condition that its affine mean curvature should be identically 0.

affine normal (**1.**) For a plane curve $\mathbf{x} = \mathbf{x}(t)$, the vector $\mathbf{x}' = \frac{d\mathbf{x}}{ds}$, where s is affine arc length.
(**2.**) For a surface $(\mathbf{x})$, the vector $\mathbf{y} = \frac{1}{2}\Delta\mathbf{x}$, where Δ is the second Beltrami operator.

affine principal normal vector For a plane curve $\mathbf{x} = \mathbf{x}(t)$, the vector $\mathbf{x}'' = \frac{d^2\mathbf{x}}{ds^2}$, where s is affine arc length.

affine surface area Let (x_1, x_2, x_3) denote the points on a surface and set

$$L = \frac{\partial^2 x_3}{\partial x_1^2}, M = \frac{\partial^2 x_3}{\partial x_1 \partial x_2}, N = \frac{\partial^2 x_3}{\partial x_2^2}.$$

The *affine surface area* is

$$\Omega = \int\int |LN - M^2|^{\frac{1}{4}} du dv.$$

affine symmetric space A complete, connected, simply connected, n-dimensional manifold M having a connection on the frame bundle such that, for every $x \in M$, the geodesic symmetry $exp_x(Z) \to exp_x(-Z)$ is the restriction to $exp_x(M_x)$ of an affine diffeomorphism of M. *See* affine diffeomorphism.

affine torsion For a space curve $\mathbf{x}(p) = \{x_1(p), x_2(p), x_3(p)\}$, the quantity

$$\tau = -\det \left\{ \begin{matrix} x_1^{(4)} & x_2^{(4)} & x_3^{(4)} \\ x_1'' & x_2'' & x_3'' \\ x_1''' & x_2''' & x_3''' \end{matrix} \right\},$$

where derivatives are with respect to affine arc length.

affine transformation (**1.**) A function of the form $f(x) = ax + b$, where a and b are constants and x is a real or complex variable. (**2.**) Members of the general linear group (invertible transformations of the form $(az + b)/(cz + d)$).

Ahlfors function *See* analytic capacity.

Ahlfors' Five Disk Theorem Let $f(z)$ be a transcendental meromorphic function, and let $A_1, A_2, \dots, A_5$ be five simply connected domains in $\mathbf{C}$ with disjoint closures. There exists $j \in \{1, 2, \dots, 5\}$ and, for any $R > 0$, a simply connected domain $D \subset \{z \in \mathbf{C} : |z| > R\}$ such that $f(z)$ is a conformal map of D onto A_j. If $f(z)$ has a finite number of poles, then 5 may be replaced by 3.

See also meromorphic function, transcendental function.

Albanese variety Let R be a Riemann surface, $H^{1,0}$ the holomorphic1, 0 forms on R, $H^{0,1*}$ its complex dual, and let a curve γ in R act on $H^{0,1}$ by integration:

$$w \to I(\gamma) = \int_\gamma w.$$

The *Albanese variety, $Alb(R)$* of R is

$$Alb(R) = H^{0,1*}/I(H_1(Z)).$$

See also Picard variety.

Alexandrov compactification For a topological space X, the set $\hat{X} = X \cup \{x\}$, for some point $x \notin X$, topologized so that the closed sets in $\hat{X}$ are (i.) the compact sets in X, and (ii.) all sets of the form $E \cup \{x\}$ where E is closed in X.

$\hat{X}$ is also called the *one point compactification* of X.

algebra of differential forms Let M be a differentiable manifold of class $\mathbf{C}^r (r \geq 1)$, $T_p(M)$ its tangent space, $T_p^*(M) = T_p(M)^*$ the dual vector space (the linear mappings from $T_p(M)$ into $\mathbf{R}$) and $T^*(M) = \cup_{p \in M} T_p^*(M)$. The *bundle of i-forms* is

$$\wedge^i (T^*(M)) = \cup_{p \in M} \wedge^i (T_p^*(M)),$$

where, for any linear map $f : V \to W$, between two vector spaces, the linear map

$$\wedge^i f : \wedge^i V \to \wedge^i W$$

is defined by $(\wedge^i f)(v_1 \wedge \cdots \wedge v_k) = f(v_1) \wedge \cdots \wedge f(v_k)$. The bundle projection is defined by $\pi(z) = p$, for $z \in \wedge^i (T_p^*(M))$.

A *differential i-form* or *differential form of degree i* is a section of the bundle of i-forms; that is, a continuous map

$$s : M \to \wedge^i (T^*(M))$$

with $\pi(s(p)) = p$. If $D^i(M)$ denotes the vector space of differential forms of degree i, the *algebra of differential forms* on M is

$$D^*(M) = \sum_{i \geq 0} \oplus D^i(M).$$

It is a graded, anticommutative algebra over $\mathbf{R}$.

algebra of sets A collection $\mathcal{F}$ of subsets of a set S such that if $E, F \in \mathcal{F}$, then (i.) $E \cup F \in \mathcal{F}$, (ii.) $E \backslash F \in \mathcal{F}$, and (iii.) $S \backslash F \in \mathcal{F}$. If $\mathcal{F}$ is also closed under the taking of countable unions, then $\mathcal{F}$ is called a σ-algebra. Algebras and σ-algebras of sets are sometimes called *fields* and *σ-fields* of sets.

algebraic analysis The study of mathematical objects which, while of an analytic nature, involve manipulations and characterizations which are algebraic, as opposed to inequalities and estimates. An example is the study of algebras of operators on a Hilbert space.

algebraic function A function $y = f(z)$ of a complex (or real) variable, which satisfies a polynomial equation

$$a_n(z)y^n + a_{n-1}(z)y^{n-1} + \ldots + a_0(z) = 0,$$

where $a_0(z), \ldots, a_n(z)$ are polynomials.

algebraic singularity *See* branch.

algebroidal function An analytic function $f(z)$ satisfying the irreducible algebraic equation

$$A_0(z)f^k + A_1(z)f^{k-1} + \cdots + A_k(z) = 0$$

with single-valued meromorphic functions $A_j(z)$ in a complex domain G is called *k-algebroidal* in G.

almost complex manifold A smooth manifold M with a field of endomorphisms J on $T(M)$ such that $J^2 = J \circ J = -I$, where I is the identity endomorphism. The field of endomorphisms is called an *almost complex structure* on M.

almost complex structure *See* almost complex manifold.

almost contact manifold An odd dimensional differentiable manifold M which admits a tensor field ϕ of type $(1, 1)$, a vector field ζ and a 1-form ω such that

$$\phi^2 X = -X + \omega(X)\zeta, \quad \omega(\zeta) = 1,$$

for X an arbitrary vector field on M. The triple (ϕ, ζ, ω) is called an *almost contact structure* on M.

almost contact structure *See* almost contact manifold.

almost everywhere Except on a set of measure 0 (applying to the truth of a proposition about points in a measure space). For example, a sequence of functions $\{f_n(x)\}$ converges *almost everywhere* to $f(x)$, provided that $f_n(x) \to f(x)$ for $x \in E$, where the complement of E has measure 0. Abbreviations are *a.e.* and *p.p.* (from the French *presque partout*).

almost periodic function in the sense of Bohr A continuous function $f(x)$ on $(-\infty, \infty)$ such that, for every $\epsilon > 0$, there is a $p = p(\epsilon) > 0$ such that, in every interval of the form $(t, t + p)$, there is at least one number τ such that $|f(x + \tau) - f(x)| \leq \epsilon$, for $-\infty < x < \infty$.

almost periodic function on a group For a complex-valued function $f(g)$ on a group G, let $f_s : G \times G \to \mathbf{C}$ be defined by $f_s(g, h) = f(gsh)$. Then f is said to be *almost periodic* if the family of functions $\{f_s(g, h) : s \in G\}$ is totally bounded with respect to the uniform norm on the complex-valued functions on $G \times G$.

almost periodic function on a topological group On a (locally compact, Abelian) group G, the uniform limit of trigonometric polynomials on G. A trigonometric polynomial is a finite linear combination of characters (i.e., homomorphisms into the multiplicative group of complex numbers of modulus 1) on G.

alpha capacity A financial measure giving the difference between a fund's actual return and its expected level of performance, given its level of risk (as measured by the beta capacity). A positive alpha capacity indicates that the fund has performed better than expected based on its beta capacity whereas a negative alpha indicates poorer performance.

alternating mapping The mapping $\mathcal{A}$, generally acting on the space of covariant tensors on a vector space, and satisfying

$$\mathcal{A}\Phi(\mathbf{v}_1, \ldots, \mathbf{v}_r) = \frac{1}{r!}\sum_{\sigma} \mathrm{sgn}\sigma\, \Phi(\mathbf{v}_{\sigma(1)}, \ldots, \mathbf{v}_{\sigma(r)}),$$

where the sum is over all permutations σ of $\{1, \ldots, r\}$.

alternating multilinear mapping A mapping $\Phi : V \times \cdots \times V \to W$, where V and W are vector spaces, such that $\Phi(v_1, \ldots, v_n)$ is linear in each variable and satisfies

$$\Phi(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n) = -\Phi(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_n).$$

alternating series A formal sum $\sum a_j$ of real numbers, where $(-1)^j a_j \geq 0$ or $(-1)^j a_{j+1} \geq 0$; i.e., the terms alternate in sign.

alternating tensor *See* antisymmetric tensor.

alternizer *See* alternating mapping.

amenable group A locally compact group G for which there is a left invariant mean on $L^\infty(G)$.

Ampere's transformation A transformation of the surface $z = f(x, y)$, defined by coordinates X, Y, Z, given by

$$X = \frac{\partial f}{\partial x}, Y = \frac{\partial f}{\partial y}, Z = \frac{\partial f}{\partial x}x + \frac{\partial f}{\partial y}y - z.$$

amplitude function For a normal lattice, let e_1, e_2, e_3 denote the stationary values of the Weierstrass $\wp$-function and, for $i = 1, 2, 3$, let $\mathbf{f}_i(u)$ be the square root of $\wp - e_i$, whose leading term at the origin is u^{-1}. Two of the Jacobi-Glaisher functions are

$$\mathrm{cs}u = \mathbf{f}_1, \mathrm{sn}u = 1/\mathbf{f}_2,$$

which are labeled in analogy with the trigonometric functions, on account of the relation $\mathrm{sn}^2 u + \mathrm{cs}^2 u = 1$. As a further part of the analogy, the *amplitude,* am u, of u, is defined to be the angle whose sine and cosine are sn$u$ and cs$u$.

amplitude in polar coordinates In polar coordinates, a point in the plane $\mathbf{R}^2$ is written (r, θ), where r is the distance from the origin and $\theta \in [0, 2\pi)$ is the angle the line segment (from the origin to the point) makes with the positive real axis. The angle θ is called the *amplitude.*

amplitude of complex number *See* argument of complex number.

amplitude of periodic function The absolute maximum of the function. For example, for the function $f(x) = A\sin(\omega x - \phi)$, the number A is the amplitude.

analysis A branch of mathematics that can be considered the foundation of calculus, arising out of the work of mathematicians such as Cauchy and Riemann to formalize the differential and integral calculus of Newton and Leibniz. *Analysis* encompasses such topics as limits, continuity, differentiation, integration, measure theory, and approximation by sequences and series, in the context of metric or more general topological spaces. Branches of analysis include real analysis, complex analysis, and functional analysis.

analysis on locally compact Abelian groups The study of the properties (inversion, etc.) of the Fourier transform, defined by

$$\hat{f}(\gamma) = \int_G f(x)(-x, \gamma)dx,$$

with respect to Haar measure on a locally compact, Abelian group G. Here $f \in L^1(G)$

and γ is a homomorphism from G to the multiplicative group of complex numbers of modulus 1. The classical theory of the Fourier transform extends with elegance to this setting.

analytic *See* analytic function.

analytic automorphism A mapping from a field with absolute value to itself, that preserves the absolute value.

See also analytic isomorphism.

analytic capacity For a compact planar set K, let $\Omega(K) = K_1 \cup \{\infty\}$, where K_1 is the unbounded component of the complement of K. Let $\mathcal{A}(K)$ denote the set of functions f, analytic on $\Omega(K)$, such that $f(\infty) = 0$ and $\|f\|_{\Omega(K)} \leq 1$. If K is not compact, $\mathcal{A}(K)$ is the union of $\mathcal{A}(E)$ for E compact and $E \subset K$. The *analytic capacity* of a planar set E is

$$\gamma(E) = \sup_{f \in \mathcal{A}(E)} |f'(\infty)|.$$

If K is compact, there is a unique function $f \in \mathcal{A}(K)$ such that $f'(\infty) = \gamma(K)$. This function f is called the *Ahlfors function* of K.

analytic continuation A function $f(z)$, analytic on an open disk $A \subset \mathbf{C}$, is a *direct analytic continuation* of a function $g(z)$, analytic on an open disk B, provided the disks A and B have nonempty intersection and $f(z) = g(z)$ in $A \cap B$.

We say $f(z)$ is an *analytic continuation* of $g(z)$ if there is a finite sequence of functions $f_1, f_2, \dots, f_n$, analytic in disks $A_1, A_2, \dots, A_n$, respectively, such that $f_1(z) = f(z)$ in $A \cap A_1$, $f_n(z) = g(z)$ in $A_n \cap B$ and, for $j = 1, \dots, n-1$, $f_{j+1}(z)$ is a direct analytic continuation of $f_j(z)$.

analytic continuation along a curve Suppose $f(z)$ is a function, analytic in a disk D, centered at z_0, $g(z)$ is analytic in a disk E, centered at z_1, and C is a curve with endpoints z_0 and z_1. We say that g is an analytic continuation of f along C, provided there is a sequence of disks $D_1, \dots, D_n$, with centers on C and an analytic function $f_j(z)$ analytic in D_j, $j = 1, \dots, n$, such that $f_1(z) = f(z)$ in $D = D_1$, $f_n(z) = g(z)$ in $D_n = E$ and, for $j = 1, \dots, n-1$, $f_{j+1}(z)$ is a direct analytic continuation of $f_j(z)$. *See* analytic continuation.

analytic curve A curve $\alpha : I \to M$ from a real interval I into an analytic manifold M such that, for any point $p_0 = \alpha(t_0)$, the chart (U_{p_0}, ϕ_{p_0}) has the property that $\phi_{p_0}(\alpha(t))$ is an analytic function of t, in the sense that $\phi_{p_0}(\alpha(t)) = \sum_{j=0}^{\infty} a_j (t-t_0)^j$ has a nonzero radius of convergence, and $a_1 \neq 0$.

analytic disk A nonconstant, holomorphic mapping $\phi : D \to \mathbf{C}^n$, were D is the unit disk in $\mathbf{C}^1$, or the image of such a map.

analytic function (**1.**) A real-valued function $f(x)$ of a real variable, is *(real) analytic* at a point $x = a$ provided $f(x)$ has an expansion in power series

$$f(x) = \sum_{j=0}^{\infty} c_j (x-a)^j,$$

convergent in some neighborhood $(a-h, a+h)$ of $x = a$.

(**2.**) A complex valued function $f(z)$ of a complex variable is *analytic* at $z = z_0$ provided

$$f'(w) = \lim_{z \to w} \frac{f(w) - f(z)}{w - z}$$

exists in a neighborhood of z_0. *Analytic in a domain* $D \subseteq \mathbf{C}$ means analytic at each point of D. Also *holomorphic, regular, regular-analytic.*

(**3.**) For a complex-valued function $f(z_1, \dots, z_n)$ of n complex variables, analytic in each variable separately.

analytic functional A bounded linear functional on $\mathcal{O}(U)$, the Fréchet space of analytic functions on an open set $U \subset \mathbf{C}^n$, with the topology of uniform convergence on compact subsets of U.

analytic geometry The study of shapes and figures, in 2 or more dimensions, with the aid of a coordinate system.

Analytic Implicit Function Theorem Suppose $F(x, y)$ is a function with a convergent power series expansion

$$F(x, y) = \sum_{j,k=0}^{\infty} a_{jk}(x - x_0)^j (y - y_0)^k,$$

where $a_{00} = 0$ and $a_{01} \neq 0$. Then there is a unique function $y = f(x)$ such that
(i.) $F(x, f(x)) = 0$ in a neighborhood of $x = x_0$;
(ii.) $f(x_0) = y_0$; and
(iii.) $f(x)$ can be expanded in a power series

$$f(x) = \sum_{j=0}^{\infty} b_j (x - x_0)^j,$$

convergent in a neighborhood of $x = x_0$.

analytic isomorphism A mapping between fields with absolute values that preserves the absolute value.

See also analytic automorphism.

analytic manifold A topological manifold with an atlas, where compatibility of two charts (U_p, ϕ_p), (U_q, ϕ_q) means that the composition $\phi_p \circ \phi_q^{-1}$ is analytic, whenever $U_p \cap U_q \neq \emptyset$. *See* atlas.

analytic neighborhood Let P be a polyhedron in the PL (piecewise linear) n-manifold M. Then an *analytic neighborhood* of P in M is a polyhedron N such that (1) N is a closed neighborhood of P in M, (2) N is a PL n-manifold, and (3) $N \downarrow P$.

analytic polyhedron Let W be an open set in $\mathbf{C}^n$ that is homeomorphic to a ball and let $f_1, \ldots, f_k$ be holomorphic on W. If the set

$$\Omega = \{z \in W : |f_j(z)| < 1, j = 1, \ldots, k\}$$

has its closure contained in W, then Ω is called an *analytic polyhedron.*

analytic set A subset A of a Polish space X such that $A = f(Z)$, for some Polish space Z and some continuous function $f : Z \to X$.

Complements of analytic sets are called *co-analytic sets.*

analytic space A topological space X (the underlying space) together with a sheaf $\mathcal{S}$, where X is locally the zero set Z of a finite set of analytic functions on an open set $D \subset \mathbf{C}^n$ and where the sections of $\mathcal{S}$ are the analytic functions on Z. Here *analytic functions on* Z (if, for example, D is a polydisk) means functions that extend to be analytic on D.

The term *complex space* is used by some authors as a synonym for *analytic space.* But sometimes, it allows a bigger class of functions as the sections of $\mathcal{S}$. Thus, while the sections of $\mathcal{S}$ are $\mathcal{H}(Z) = \mathcal{H}(D)/\mathcal{I}(Z)$ (the holomorphic functions on D modulo the ideal of functions vanishing on Z) for an analytic space, $\mathcal{H}(Z)$ may be replaced by $\hat{\mathcal{H}}(Z) = \mathcal{H}(D)/\hat{\mathcal{I}}$, for a complex space, where $\hat{\mathcal{I}}$ is some other ideal of $\mathcal{H}(D)$ with zero set Z.

angle between curves The angle between the tangents of two curves. *See* tangent line.

angular derivative Let $f(z)$ be analytic in the unit disk $D = \{z : |z| < 1\}$. Then f has an *angular derivative* $f'(\zeta)$ at $\zeta \in \partial D$ provided

$$f'(\zeta) = \lim_{r \to 1-} f'(r\zeta).$$

antiderivative A function $F(x)$ is an *antiderivative* of $f(x)$ on a set $S \subset \mathbf{R}$, provided F is differentiable and $F'(x) = f(x)$, on S. Any two antiderivatives of $f(x)$ must differ by a constant (if S is connected) and so, if $F(x)$ is one antiderivative of f, then any antiderivative has the form $F(x) + C$, for some real constant C. The usual notation for the most general antiderivative of f is

$$\int f(x)dx = F(x) + C.$$

antiholomorphic mapping A mapping whose complex conjugate, or adjoint, is analytic.

antisymmetric tensor A covariant tensor Φ of order r is *antisymmetric* if, for each $i, j, 1 \le i, j \le r$, we have

$$\Phi(\mathbf{v_1}, \dots, \mathbf{v_i}, \dots, \mathbf{v_j}, \dots, \mathbf{v_r}) = -\Phi(\mathbf{v_1}, \dots, \mathbf{v_j}, \dots, \mathbf{v_i}, \dots, \mathbf{v_r}).$$

Also called an *alternating,* or *skew* tensor, or an *exterior form.*

Appell hypergeometric function An extension of the hypergeometric function to two variables, resulting in four kinds of functions (Appell 1925):

$$G_1(a; b, c; d; x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_m(c)_n}{m!n!(d)_{m+n}} x^m y^n$$

$$G_2(a; b, c; d, d'; x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_m(c)_n}{m!n!(d)_m(d')_n} x^m y^n$$

$$G_3(a, a'; b, c'; d; x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_m(a')_n(b)_m(c)_n}{m!n!(d)_{m+n}} x^m y^n$$

$$G_4(a; b; d, d'; x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{m!n!(d)_m(d')_n} x^m y^n.$$

Appell defined these functions in 1880, and Picard showed in 1881 that they can be expressed by integrals of the form

$$\int_0^1 u^a(1-u)^b(1-xu)^d(1-yu)^q\, du.$$

approximate derivative *See* approximately differentiable function.

approximate identity On $[-\pi, \pi]$, a sequence of functions $\{e_j\}$ such that
(i.) $e_j \ge 0$, $j = 1, 2, \dots$;
(ii.) $1/2\pi \int_{-\pi}^{\pi} e_j(t)dt = 1$;
(iii.) for every ϵ with $\pi > \epsilon > 0$,

$$\lim_{j\to\infty} \int_{-\epsilon}^{\epsilon} e_j(t)dt = 0.$$

approximately differentiable function A function $F : [a, b] \to \mathbf{R}$ (at a point $c \in [a, b]$) such that there exists a measurable set $E \subseteq [a, b]$ such that $c \in E$ and is a density point of E and $F|_E$ is differentiable at c. The *approximate derivative* of F at c is the derivative of $F|_E$ at c.

approximation (**1.**) An approximation to a number x is a number that is close to x. More precisely, given an $\epsilon > 0$, an approximation to x is a number y such that $|x - y| < \epsilon$. We usually seek an approximation to x from a specific class of numbers. For example, we may seek an approximation of a real number from the class of rational numbers.
(**2.**) An approximation to a function f is a function that is close to f in some appropriate measure. More precisely, given an $\epsilon > 0$, an approximation to f is a function g such that $\|f - g\| < \epsilon$ for some norm $\|\cdot\|$. We usually seek an approximation to f from a specific class of functions. For example, for a continuous function f defined on a closed interval I we may seek a polynomial g such that $\sup_{x\in I} |f(x) - g(x)| < \epsilon$.

arc length (**1.**) For the graph of a differentiable function $y = f(x)$, from $x = a$ to $x = b$, in the plane, the integral

$$\int_a^b \sqrt{1 + (\frac{dy}{dy})^2}dx.$$

(**2.**) For a curve $t \to p(t), a \le t \le b$, of class $\mathbf{C}^1$, on a Riemannian manifold with inner product $\Phi(X_p, Y_p)$ on its tangent space at p, the integral

$$\int_a^b \left(\Phi(\frac{dp}{dt}, \frac{dp}{dt})\right)^{\frac{1}{2}} dt.$$

Argand diagram The representation $z = re^{i\theta}$ of a complex number z.

argument function The function $\arg(z) = \theta$, where z is a complex number with the representation $z = re^{i\theta}$, with r real and non-negative. The choice of θ is, of course, not unique and so $\arg(z)$ is not a function without further restrictions such as $-\pi < \arg(z) \le \pi$ (principal argument) or the requirement that it be continuous, together with a specification of the value at some point.

argument of complex number The angle θ in the representation $z = re^{i\theta}$ of a complex number z. Also *amplitude.*

argument of function The domain variable; so that if $y = f(x)$ is the function assigning the value y to a given x, then x is the *argument* of the function f. Also *independent variable.*

argument principle Let $f(z)$ be analytic on and inside a simple closed curve $C \subset \mathbf{C}$, except for a finite number of poles inside C, and suppose $f(z) \neq 0$ on C. Then $\Delta \arg f$, the net change in the argument of f, as z traverses C, satisfies $\Delta \arg f = N - P$, the number of zeros minus the number of poles of f inside C.

arithmetic mean For n real numbers, $a_1, a_2, \dots, a_n$, the number $\frac{a_1+a_2+\ldots+a_n}{n}$. For a real number r, the *arithmetic mean of order* r is

$$\frac{\sum_{j=1}^{n}(r+1)\cdots(r+n-j)a_j/(n-j)!}{\sum_{j=1}^{n}(r+1)\cdots(r+n-j)/(n-j)!}.$$

arithmetic progression A sequence $\{a_j\}$ where a_j is a linear function of j: $a_j = cj + r$, with c and r independent of j.

arithmetic-geometric mean The *arithmetic-geometric mean* (AGM) $M(a, b)$ of two numbers a and b is defined by starting with $a_0 \equiv a$ and $b_0 \equiv b$, then iterating

$$a_{n+1} = \tfrac{1}{2}(a_n + b_n) \quad b_{n+1} = \sqrt{a_n b_n}$$

until $a_n = b_n$. The sequences a_n and b_n converge toward each other, since

$$a_{n+1} - b_{n+1} = \tfrac{1}{2}(a_n + b_n) - \sqrt{a_n b_n} = \frac{a_n - 2\sqrt{a_n b_n} + b_n}{2}.$$

But $\sqrt{b_n} < \sqrt{a_n}$, so

$$2b_n < 2\sqrt{a_n b_n}.$$

Now, add $a_n - b_n - 2\sqrt{a_n b_n}$ so each side

$$a_n + b_n - 2\sqrt{a_n b_n} < a_n - b_n,$$

so

$$a_{n+1} - b_{n+1} < \tfrac{1}{2}(a_n - b_n).$$

The AGM is useful in computing the values of complete elliptic integrals and can also be used for finding the inverse tangent. The special value $1/M(1, \sqrt{2})$ is called Gauss's constant.

The AGM has the properties

$$\begin{aligned}
\lambda M(a, b) &= M(\lambda a, \lambda b) \\
M(a, b) &= M\left(\tfrac{1}{2}(a+b), \sqrt{ab}\right) \\
M(1, \sqrt{1-x^2}) &= M(1+x, 1-x) \\
M(1, b) &= \frac{1+b}{2} M\left(1, \frac{2\sqrt{b}}{1+b}\right).
\end{aligned}$$

The Legendre form is given by

$$M(1, x) = \prod_{n=0}^{\infty} \tfrac{1}{2}(1 + k_n),$$

where $k_0 \equiv x$ and

$$k_{n+1} \equiv \frac{2\sqrt{k_n}}{1 + k_n}.$$

Solutions to the differential equation

$$(x^3 - x)\frac{d^2y}{dx^2} + (3x^2 - 1)\frac{dy}{dx} + xy = 0$$

are given by $[M(1 + x, 1 - x)]^{-1}$ and $[M(1, x)]^{-1}$.

A generalization of the arithmetic-geometric mean is

$$I_p(a,b) = \int_0^\infty \frac{x^{p-2}dx}{(x^p+a^p)^{1/p}(x^p+b^p)^{(p-1)/p}},$$

which is related to solutions of the differential equation

$$x(1-x^p)Y'' + [1-(p+1)x^p]Y' - (p-1)x^{p-1}Y = 0.$$

When $p = 2$ or $p = 3$, there is a modular transformation for the solutions of the above equation that are bounded as $x \to 0$. Letting $J_p(x)$ be one of these solutions, the transformation takes the form

$$J_p(\lambda) = \mu J_p(x),$$

where

$$\lambda = \frac{1-u}{1+(p-1)u} \quad \mu = \frac{1+(p-1)u}{p}$$

and

$$x^p + u^p = 1.$$

The case $p = 2$ gives the arithmetic-geometric mean, and $p = 3$ gives a cubic relative discussed by Borwein and Borwein (1990, 1991) and Borwein (1996) in which, for $a, b > 0$ and $I(a,b)$ defined by

$$I(a,b) = \int_0^\infty \frac{t\,dt}{[(a^3+t^3)(b^3+t^3)^2]^{1/3}},$$

$$I(a,b) = I\left(\frac{a+2b}{3}, \left[\frac{b}{3}(a^2+ab+b^2)\right]\right).$$

For iteration with $a_0 = a$ and $b_0 = b$ and

$$a_{n+1} = \frac{a_n+2b_n}{3}$$

$$b_{n+1} = \frac{b_n}{3}(a_n^2 + a_nb_n + b_n^2),$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \frac{I(1,1)}{I(a,b)}.$$

Modular transformations are known when $p = 4$ and $p = 6$, but they do not give identities for $p = 6$ (Borwein 1996).

See also arithmetic-harmonic mean.

arithmetic-harmonic mean For two given numbers a, b, the number $A(a,b)$, obtained by setting $a_0 = a, b_0 = b$, and, for $n \geq 0$, $a_{n+1} = \frac{1}{2}(a_n + b_n)$, $b_{n+1} = 2a_nb_n/(a_n + b_n)$ and $A(a,b) = \lim_{n\to\infty} a_n$. The sequences a_n and b_n converge to a common value, since $a_n - b_n \leq \frac{1}{2}(a_{n-1} - b_{n-1})$, if a, b are nonnegative, and we have $A(a_0, b_0) = \lim_{n\to\infty} a_n = \lim b_n = \sqrt{ab}$, which is just the geometric mean.

Arzela-Ascoli Theorem The theorem consists of two theorems:

Propagation Theorem. If $\{f_n(x)\}$ is an equicontinuous sequence of functions on $[a,b]$ such that $\lim_{n\to\infty} f_n(x)$ exists on a dense subset of $[a,b]$, then $\{f_n\}$ is uniformly convergent on $[a,b]$.

Selection Theorem. If $\{f_n(x)\}$ is a uniformly bounded, equicontinuous sequence on $[a,b]$, then there is a subsequence which is uniformly convergent on $[a,b]$.

associated radii of convergence Consider a power series in n complex variables: $\sum a_{i_1 i_2 \dots i_n} z_1^{i_1} z_2^{i_2} \dots z_n^{i_n}$. Suppose $r_1, r_2, \dots, r_n$ are such that the series converges for $|z_1| < r_1, |z_2| < r_2, \dots, |z_n| < r_n$ and diverges for $|z_1| > r_1, |z_2| > r_2, \dots, |z_n| > r_n$. Then $r_1, r_2, \dots, r_n$ are called *associated radii of convergence.*

astroid A hypocycloid of four cusps, having the parametric equations

$$x = 4a\cos^3 t,\ y = 4a\sin^3 t.$$

$(-\pi \leq t \leq \pi)$. The Cartesian equation is

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

asymptote For the graph of a function $y = f(x)$, either (i.) a *vertical asymptote:* a vertical line $x = a$, where $\lim_{x\to a} f(x) = \infty$; (ii.) a *horizontal asymptote:* a horizontal line $y = a$ such that $\lim_{x\to\infty} f(x) = a$; or (iii.)

an *oblique asymptote:* a line $y = mx + b$ such that $\lim_{x\to\infty}[f(x) - mx - b] = 0$.

asymptotic curve Given a regular surface M, an *asymptotic curve* is formally defined as a curve $\mathbf{x}(t)$ on M such that the normal curvature is 0 in the direction $\mathbf{x}'(t)$ for all t in the domain of $\mathbf{x}$. The differential equation for the parametric representation of an asymptotic curve is

$$eu'^2 + 2fu'v' + gv'^2 = 0,$$

where e, f, and g are second fundamental forms. The differential equation for asymptotic curves on a Monge patch $(u, v, h(u, v))$ is

$$h_{uu}u'^2 + 2h_{uu}u'v' + h_{vv}v'^2 = 0,$$

and on a polar patch $(r\cos\theta, 4\sin\theta, h(r))$ is

$$h''(r)r'^2 + h'(r)r\theta'^2 = 0.$$

asymptotic direction A unit vector X_p in the tangent space at a point p of a Riemannian manifold M such that $(S(X_p), X_p) = 0$, where S is the shape operator on $T_p(M)$: $S(X_p) = -(d\mathbf{N}/dt)_{t=0}$.

asymptotic expansion A divergent series, typically one of the form

$$\sum_{j=0}^{\infty} \frac{A_j}{z^j},$$

is an *asymptotic expansion* of a function $f(z)$ for a certain range of z, provided the remainder $R_n(z) = z^n[f(z) - s_n(z)]$, where $s_n(z)$ is the sum of the first $n + 1$ terms of the above divergent series, satisfies

$$\lim_{|z|\to\infty} R_n(z) = 0$$

(n fixed) although

$$\lim_{n\to\infty} |R_n(z)| = \infty$$

(z fixed).

asymptotic path A path is a continuous curve. *See also* asymptotic curve.

asymptotic power series *See* asymptotic series.

asymptotic rays Let M be a complete, open Riemannian manifold of dimension ≥ 2. A geodesic $\gamma : [0, \infty) \rightarrow M$, emanating from p and parameterized by arc length, is called a *ray emanating from* p if $d(\gamma(t), \gamma(s)) = |t - s|$, for $t, s \in [0, \infty)$. Two rays, γ, γ' are *asymptotic* if $d(\gamma(t), \gamma'(t)) \leq |t - s|$ for all $t \geq 0$.

asymptotic sequence Let R be a subset of $\mathbf{R}$ or $\mathbf{C}$ and c a limit point of R. A sequence of functions $\{f_j(z)\}$, defined on R, is called an *asymptotic sequence* or *scale* provided

$$f_{j+1}(z) = o(f_j(z))$$

as $z \rightarrow c$ in R, in which case we write the asymptotic series

$$f(z) \sim \sum_{j=0}^{\infty} a_j f_j(z) \qquad (z \rightarrow c, \text{ in } R)$$

for a function $f(z)$, whenever, for each n,

$$f(z) = \sum_{j=0}^{n-1} a_j f_j(z) + O(f_n(z)),$$

as $z \rightarrow c$ in R.

asymptotic series *See* asymptotic sequence.

asymptotic stability Given an autonomous differential system $y' = f(y)$, where $f(y)$ is defined on a set containing $y = 0$ and satisfies $f(0) = 0$, we say the solution $y \equiv 0$ is *asymptotically stable, in the sense of Lyapunov*, if
(i.) for every $\epsilon > 0$, there is a $\delta_\epsilon > 0$ such that, if $|y_0| < \delta_\epsilon$, then there is a solution $y(t)$ satisfying $y(0) = y_0$ and $|y(t)| < \epsilon$, for $t \geq 0$; and
(ii.) $y(t) \rightarrow 0$, as $t \rightarrow \infty$.

Whenever (i.) is satisfied, the solution $y \equiv 0$ is said to be *stable, in the sense of Lyapunov.*

asymptotic tangent line A direction of the tangent space $T_p(S)$ (where S is a regular surface and $p \in S$) for which the normal curvature is zero.

See also asymptotic curve, asymptotic path.

Atiyah-Singer Index Theorem A theorem which states that the analytic and topological indices are equal for any elliptic differential operator on an n-dimensional compact differentiable $\mathbf{C}^\infty$ boundaryless manifold.

atlas By definition, a topological space M is a differentiable [resp., $\mathbf{C}^\infty$, analytic] manifold if, for every point $p \in M$, there is a neighborhood U_p and a homeomorphism ϕ_p from U_p into $\mathbf{R}^n$. The neighborhood U_p or, sometimes, the pair (U_p, ϕ_p), is called a *chart.* Two charts U_p, U_q are required to be *compatible*; i.e., if $U_p \cap U_q \neq \emptyset$ then the functions $\phi_p \circ \phi_q^{-1}$ and $\phi_q \circ \phi_p^{-1}$ are differentiable [resp, $\mathbf{C}^\infty$, analytic]. The set of all charts is called an *atlas*. An atlas $\mathcal{A}$ is *complete* if it is maximal in the sense that if a pair U, ϕ is compatible with one of the U_p, ϕ_p in $\mathcal{A}$, then U belongs to $\mathcal{A}$.

In the case of a differentiable [resp., $\mathbf{C}^\infty$, analytic] manifold *with boundary,* the maps ϕ_p may map from U_p to either $\mathbf{R}^n$ or $\mathbf{R}^n_+ = \{(x_1, \ldots, x_n) : x_j \geq 0, \text{ for } j = 1, \ldots, n\}$.

atom For a measure μ on a set X, a point $x \in X$ such that $\mu(x) > 0$.

automorphic form Let G be a Kleinian group acting on a domain $D \subset \mathbf{C}$ and q a positive integer. A measurable function $\sigma : D \to \mathbf{C}$ is a *measurable automorphic form of weight $-2q$ for G* if

$$(\sigma \circ g)(g')^q = \sigma$$

almost everywhere on D, for all $g \in G$.

automorphic function A meromorphic function $f(z)$ satisfying $f(Tz) = f(z)$ for T belonging to some group of linear fractional transformations (that is, transformations of the form $Tz = (az+b)/(cz+d)$). When the linear fractional transformations come from a subgroup of the modular group, f is called a *modular* function.

autonomous linear system *See* autonomous system.

autonomous system A system of differential equations $\frac{d\mathbf{y}}{dt} = \mathbf{f}(y)$, where $\mathbf{y}$ and $\mathbf{f}$ are column vectors, and $\mathbf{f}$ is independent of t.

auxiliary circle Suppose a central conic has center of symmetry P and foci F and F', each at distance a from P. The circle of radius a, centered at P, is called the *auxiliary circle.*

axiom of continuity One of several axioms defining the real number system uniquely: Let $\{x_j\}$ be a sequence of real numbers such that $x_1 \leq x_2 \leq \ldots$ and $x_j \leq M$ for some M and all j. Then there is a number $L \leq M$ such that $x_j \to L$, $j \to \infty$ and $x_j \leq L$, $j = 1, 2, \ldots$.

This axiom, together with axioms determining addition, multiplication, and ordering serves to define the real numbers uniquely.

axis (**1.**) The Cartesian coordinates of a point in a plane are the directed distances of the point from a pair of intersecting lines, each of which is referred to as an axis. In three-dimensional space, the coordinates are the directed distances from coordinate planes; an *axis* is the intersection of a pair of coordinate planes.

(**2.**) If a curve is symmetric about a line, then that line is known as an *axis* of the curve. For example, an ellipse has two axes: the major axis, on which the foci lie, and a minor axis, perpendicular to the major axis through the center of the ellipse.

(**3.**) The *axis* of a surface is a line of sym-

metry for that surface. For example, the axis of a right circular conical surface is the line through the vertex and the center of the base. The axis of a circular cylinder is the line through the centers of the two bases.
(**4.**) In polar coordinates (r, θ), the polar axis is the ray that is the initial side of the angle θ.

axis of absolute convergence *See* abscissa of absolute convergence.

axis of convergence *See* abscissa of convergence.

axis of regularity *See* abscissa of regularity.

axis of rotation A surface of revolution is obtained by rotating a curve in the plane about a line in the plane that has the curve on one side of it. This line is referred to as the *axis of rotation* of the surface.

B

Baire σ-algebra The smallest σ-algebra on a compact Hausdorff space X making all the functions in $\mathbf{C}(X)$ measurable. The sets belonging to the Baire σ-algebra are called the *Baire subsets* of X.

Baire Category Theorem A nonempty, complete metric space is of the second category. That is, it cannot be written as the countable union of nowhere dense subsets.

Baire function A function that is measurable with respect to the ring of Baire sets. Also *Baire measurable function.*

Baire measurable function *See* Baire function.

Baire measure A measure on a Hausdorff space X, for which all the Baire subsets of X are measurable and which is finite on the compact G_δ sets.

Baire property A subset A of a topological space has the *Baire property* if there is a set B of the first category such that $(A \backslash B) \cup (B \backslash A)$ is open.

Baire set *See* Baire σ-algebra.

balanced set A subset M of a vector space V over $\mathbf{R}$ or $\mathbf{C}$ such that $\alpha x \in M$, whenever $x \in M$ and $|\alpha| \leq 1$.

Banach algebra A vector space B, over the complex numbers, with a multiplication defined and satisfying (for $x, y, z \in B$)
(i.) $x \cdot y = y \cdot x$;
(ii.) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$;
(iii.) $x \cdot (y + z) = x \cdot y + x \cdot z$;
and, in addition, with a norm $\| \cdot \|$ making B into a Banach space and satisfying
(iv.) $\|x \cdot y\| \leq \|x\| \|y\|$, for $x, y \in B$.

Banach analytic space A Banach space of analytic functions. (*See* Banach space.) Examples are the Hardy spaces. *See* Hardy space.

Banach area Let $T : A \to \mathbf{R}^3$ be a continuous mapping defining a surface in $\mathbf{R}^3$ and let K be a polygonal domain in A. Let P_0 be the projection of $\mathbf{R}^3$ onto a plane E and let m denote Lebesgue measure on $PT(K)$. The *Banach area* of $T(A)$ is

$$\sup_S \sum_{K \in S} [m^2(A_1) + m^2(A_2) + m^2(A_3)]$$

where A_j are the projections of K onto coordinate planes in $\mathbf{R}^3$ and S is a finite collection of non-overlapping polygonal domains in A.

Banach manifold A topological space M such that every point has a neighborhood which is homeomorphic to the open unit ball in a Banach space.

Banach space A complete normed vector space. That is, a vector space X, over a scalar field ($\mathbf{R}$ or $\mathbf{C}$) with a nonnegative real valued function $\| \cdot \|$ defined on X, satisfying (i.) $\|cx\| = |c| \|x\|$, for c a scalar and $x \in X$; (ii.) $\|x\| = 0$ only if $x = 0$, for $x \in X$; and (iii.) $\|x + y\| \leq \|x\| + \|y\|$, for $x, y \in X$. In addition, with the metric $d(x, y) = \|x - y\|$, X is assumed to be complete.

Banach-Steinhaus Theorem Let X be a Banach space, Y a normed linear space and $\{\Lambda_\alpha : X \to Y\}$, a family of bounded linear mappings, for $\alpha \in A$. Then, either there is a constant $M < \infty$ such that $\|\Lambda_\alpha\| \leq M$, for all $\alpha \in A$, or $\sup_{\alpha \in A} \|\Lambda_\alpha x\| = \infty$, for all x in some subset $S \subset X$, which is a dense G_δ.

Barnes's extended hypergeometric function Let $G(a, b; c; z)$ denote the sum of the hypergeometric series, convergent for

$|z| < 1$:

$$\sum_{j=0}^{\infty} \frac{\Gamma(a+j)\Gamma(b+j)}{\Gamma(c+j)j!} z^j,$$

which is the usual hypergeometric function $F(a, b; c; z)$ divided by the constant $\Gamma(c)/[\Gamma(a)\Gamma(b)]$. Barnes showed that, if $|\arg(-z)| < \pi$ and the path of integration is curved so as to lie on the right of the poles of $\Gamma(a+\zeta)\Gamma(b+\zeta)$ and on the left of the poles of $\Gamma(-\zeta)$, then

$$G(a, b; c; z) =$$

$$\frac{1}{2\pi i} \int_{-\pi i}^{\pi i} \frac{\Gamma(a+\zeta)\Gamma(b+\zeta)\Gamma(-\zeta)}{\Gamma(c+\zeta)} (-z)^{\zeta} d\zeta,$$

thus permitting an analytic continuation of $F(a, b; c; z)$ into $|z| > 1$, $\arg(-z) < \pi$.

barrel A convex, balanced, absorbing subset of a locally convex topological vector space. *See* balanced set, absorbing.

barrel space A locally convex topological vector space, in which every barrel is a neighborhood of 0. *See* barrel.

barrier *See* branch.

barycentric coordinates Let $p_0, p_1, \ldots, p_n$ denote points in $\mathbf{R}^n$, such that $\{p_j - p_0\}$ are linearly independent. Express a point $P = (a_1, a_2, \ldots, a_n)$ in $\mathbf{R}^n$ as

$$P = \sum_{j=0}^{n} \mu_j p_j$$

where $\sum_0^n \mu_j = 1$ (this can be done by expressing P as a linear combination of $p_1 - p_0, p_2 - p_0, \ldots, p_n - p_0$). The numbers $\mu_0, \mu_1, \ldots, \mu_n$ are called the *barycentric coordinates* of the point P. The point of the terminology is that, if $\{\mu_0, \ldots, \mu_n\}$ are nonnegative weights of total mass 1, assigned to the points $\{p_0, \ldots, p_n\}$, then the point $P = \sum_0^n \mu_j p_j$ is the center of mass or *barycenter* of the $\{p_j\}$.

basic vector field Let M, N be Riemannian manifolds and $\pi : M \to N$ a Riemannian submersion. A horizontal vector field X on M is called *basic* if there exists a vector field $\hat{X}$ on N such that $D\pi(p)X_p = \hat{X}_{\pi(p)}$, for $p \in M$.

basis A finite set $\{x_1, \ldots, x_n\}$, in a vector space V such that (i.) $\{x_j\}$ is linearly independent, that is, $\sum_{j=1}^n c_j x_j = 0$ only if $c_1 = c_2 = \ldots = c_n = 0$, and (ii.) every vector $v \in V$ can be written as a linear combination $v = \sum_{j=1}^n c_j x_j$.

An *infinite* set $\{x_j\}$ satisfying (i.) (for every n) and (ii.) (for some n) is called a *Hamel basis.*

BDF *See* Brown-Douglas-Fillmore Theorem.

Bell numbers The number of ways a set of n elements can be partitioned into nonempty subsets, denoted B_n. For example, there are five ways the numbers $\{1, 2, 3\}$ can be partitioned: $\{\{1\}, \{2\}, \{3\}\}$, $\{\{1, 2\}, \{3\}\}$, $\{\{1, 3\}, \{2\}\}$, $\{\{1\}, \{2, 3\}\}$, and $\{\{1, 2, 3\}\}$, so $B_3 = 5$. $B_0 = 1$ and the first few Bell numbers for $n = 1, 2, \ldots$ are 1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975,.... *Bell numbers* are closely related to Catalan numbers.

The integers B_n can be defined by the sum

$$B_n = \sum_{k=1}^{n} S(n, k),$$

where $S(n, k)$ is a Stirling number of the second kind, or by the generating function

$$e^{e^n - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n.$$

Beltrami equation The equation $D_\ell f = 0$. *See* Beltrami operator.

Beltrami operator Given by

$$D_\ell = \sum_{i=1}^{\ell} x_i^2 \frac{\partial^2}{\partial x_i^2} + \sum_{i \neq j} \frac{x_j^2}{x_i - x_j} \frac{\partial}{\partial x_j}.$$

The Beltrami operator appears in the expansions in many distributions of statistics based on normal populations.